Advanced Mathematical Models & Applications
Vol.7, No.3, 2022, pp.253-263

A MODIFICATION ON THE NEW GENERAL INTEGRAL TRANSFORM*

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Abstract. In this paper, we present a modified version of the new general integral transform which is given by the first author (Journal of Advanced Research, Vol.32, pp. 133-138, 2021) and find out its relationship with Laplace type and some other integral transformations. Further, we derive the inversion formula, generalized convolution theorem for the same. As an application of it we solved ordinary differential equation as well as Volterra integral equation.

Keywords: Integral transform, ordinary differential equations, convolution.
AMS Subject Classification: 44A10, 33E30, 44A35.
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Received: 13 September 2022; Revised: 11 November 2022; Accepted: 18 November 2022; Published: 15 December 2022.

1 Introduction

The idea of Laplace type integral transformations can be utilized of converting one form of problems into another form of problem which is rather simpler to solve and then using inversion formula coming back to the solution into original form Ansari (2012). In recent era Caputo & Fabrizio (2016); Podlubny (1999) most of the linear / Nonlinear boundary value problems, Initial value problems are effectually solved by Laplace type integral transformations which includes Laplace, Mellin, Fourier, Elzaki etc. transformations having applications mostly in daily life and divisions of science like financial mathematics, artificial engineering, quantum calculus, Bio- Engineering, CFD, Abel's integral equations, Bio-mathematics.

In 2021, the first author introduced a new general integral transform in the class of Laplace transform Jafari (2021) which is called later as Jafari transform and Jafari-Yang transform (El-Mesady et al., 2021; Meddahi et al., 2022). In this paper, we present a modification of this new general integral transformation (in view of Ansari (2012); Manjarekar (2017)) . Also we investigate its relationship with the other transformations in generalized way. Moreover, as an application of the said transform we have solved ordinary differential equation and Volterra integral equation as a part of modified generalized integral transformation definition.

The paper mainly divided into four parts, in the first part we have given some basic definitions, in the second part we have given the definition of modified generalized integral transformation and given generalized definition of convolution along with inversion, derivative and convolution property. In the third part, we have provided the relationship of it with other Integral transformations including Laplace type integral transforms, Mellin transform, L_2 - transform,

*How to cite (APA)**: Jafari, H., Manjarekar, S. (2022). A modification on the new general integral transform. *Advanced Mathematical Models & Applications*, *7*(3), 253-263.

Abel's transform, Laplace-Carson and Laplace-Stieltjes transform. The fourth part contains an application of modified generalized integral transformation towards ordinary differential equations and Volterra integral equations along with the discussion of the obtained result conclusion part ends our research article.

2 Basic definition

In the section we present some basic definitions needed in proving the main results.

Definition 1. *(Ansari, 2012) If function $f(t)$ is continuous piecewise and is of exponential order then its Laplace – type integral transform is given by:*

$$L_\epsilon\{f(t);p\} = \int_0^\infty \epsilon'(t)e^{-\phi(p)\epsilon(t)}dt. \tag{1}$$

In the above definition, $\phi(p)$ is a function which is invertible such that $\epsilon(t) = \int e^{-a(t)}dt$ is exponential function and $a(t)$ is a function which also invertible.

Definition 2. *(Jafari, 2021) If a function $f(t)$ which is to be integrable and defined for $t \geq 0$ and $p(s) \neq 0$ and $q(s)$ are positive real valued function then its new generalized integral transform is given by*

$$T\{f(t);s\} = F(s) = p(s)\int_0^\infty f(t)e^{-q(s)t}dt. \tag{2}$$

Provided the integral exist for some $q(s)$.

Definition 3. *(Manjarekar, 2017) The Generalized Elzaki -Tarig Transform of a function $f(t)$ which is continuous in piecewise sense and exponential order is given by*

$$\Im_\epsilon\{f(t);s\} = \int_0^\infty \phi(\frac{1}{s})\phi_1(s)\epsilon'(t)f(t)e^{-\phi(s)\epsilon(t)}dt. \tag{3}$$

Provided $f(t)$ satisfies the conditions in Manjarekar (2017).

3 Main Result

Definition 4. *The modified generalized integral transform can be defining by using the Definitions 1 to 3 as*

$$\Im_{\epsilon(t)}\{f(t);s\} = \int_0^\infty p(s)\epsilon'(t)f(t)e^{-q(s)\epsilon(t)}dt, \tag{4}$$

provided $f(t)$ satisfies the conditions in Manjarekar (2017) and $p(s)$, $q(s)$ are invertible functions of s alongwith $\epsilon(t) = \int e^{-a(t)}dt$ is exponential function and $a(x)$ is function which is invertible and the integral is bounded.

Definition 5. *[Generalized Convolution] Given two function $f(t)$ and $g(t)$ then there generalized convolution at $\epsilon(t)$ is defined by*

$$[f * g][\epsilon(t)] = \int_0^t \epsilon'(\tau)f(\epsilon(t) - \epsilon(\tau))g(\epsilon(\tau))d\tau. \tag{5}$$

For $\epsilon(t) = t$ it coincides with the original definition of convolution.

Theorem 1 (**Inversion Formula**). *If $f(t)$ satisfies the condition in Manjarekar (2017) then its inverse integral transformation is given by*

$$f(t) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \Im_{\epsilon(t)}\{f(t);s\}e^{q(s)\epsilon(t)}ds. \tag{6}$$

Proof. The definition of modified general integral transform (4), it tells us that

$$\Im_{\epsilon(t)}\{f(t);s\} = \int_0^\infty p(s)\epsilon'(t)f(t)e^{-q(s)\epsilon(t)}dt,$$

satisfies the conditions in 4 then its inverse transform is defined using the relationship Manjarekar (2017) as;

$$f(t) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \Im_{\epsilon(t)}\{f(t);s\}e^{q(s)\epsilon(t)}ds.$$

□

Theorem 2 (**First Derivative Property).** *Let $f(t)$ is a function satisfies the conditions in Manjarekar (2017) with $f(t)$ is differentiable function with $p(s)$ and $q(s)$ are positive real valued functions then;*

$$\Im_{\epsilon(t)}\{f'(t);s\} = q(s)[\Im_{\epsilon(t)}\{\epsilon'(t)f(t);s\}] - p(s)\int_0^\infty \epsilon''(t)f(t)e^{-q(s)\epsilon(t)}dt - p(s)f(0)\epsilon'(0^+)e^{-q(s)\epsilon(0^+)}.$$

Proof. By definition of modified general integral transform (4), we have

$$\Im_{\epsilon(t)}\{f'(t);s\} = \int_0^\infty p(s)\epsilon'(t)f'(t)e^{-q(s)\epsilon(t)}dt.$$

Applying integration by part to the above eqation; gives us

$$\Im_{\epsilon(t)}\{f'(t);s\} = p(s)[|\epsilon'(t)f(t)e^{-q(s)\epsilon(t)}|_0^\infty - q(s)\int_0^\infty \frac{d}{dt}[\epsilon(t)e^{-q(s)\epsilon(t)}]f(t)dt].$$

After rearranging the terms and using the definition (4) it resulted into;

$$\Im_{\epsilon(t)}\{f'(t);s\} = q(s)[\Im_{\epsilon(t)}\{\epsilon'(t)f(t);s\}] - p(s)\int_0^\infty \epsilon''(t)f(t)e^{-q(s)\epsilon(t)}dt - p(s)f(0)\epsilon'(0^+)e^{-q(s)\epsilon(0^+)},$$

which coincides with Laplace transform of derivative property by taking $p(s) = 1$ and $q(s) = s$ with $\epsilon(t) = t$ □

Theorem 3 (**Convolution).** *Let $F_{\epsilon(t)}(s)$ and $G_{\epsilon(t)}(s)$ are modified general integral transform of two functions $f(t)$ and $g(t)$ respectively. Then the convolution theorem is defined as,*

$$\Im_{\epsilon(t)}\{[f*g](\epsilon(t));s\} = \frac{1}{p(s)}F_{\epsilon(t)}(s)G_{\epsilon(t)}(s),$$

where $p(s) \neq 0$.

Proof. Using the definition (4) and convolution we have,

$$F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = [\int_0^\infty p(s)\epsilon'(x)f(x)e^{-q(s)\epsilon(x)}dx][\int_0^\infty p(s)\epsilon'(\tau)g(\tau)e^{-q(s)\epsilon(\tau)}d\tau].$$

This can be written as iterated integral form as,

$$F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = [p(s)]^2\int_0^\infty\int_0^\infty \epsilon'(x)\epsilon'(\tau)e^{-q(s)[\epsilon(x)+\epsilon(\tau)]}f(x)g(\tau)dxd\tau.$$

By changing the variable $\epsilon(x) = \epsilon(t) - \epsilon(\tau) \Rightarrow \epsilon'(x)dx = \epsilon'(t)dt$, which gives us after change of order of integration as,

$$F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = [p(s)]^2\int_0^\infty\int_{\epsilon(\tau)}^\infty \epsilon'(t)\epsilon'(\tau)e^{-q(s)[\epsilon(t)]}f(\epsilon(t)-\epsilon(\tau))g(\epsilon(\tau))dtd\tau$$

$$\Rightarrow F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = p(s)\left[p(s)\int_0^\infty \epsilon'(t)e^{-q(s)[\epsilon(t)]}\Big[\int_0^t \epsilon'(\tau)f(\epsilon(t)-\epsilon(\tau))g(\epsilon(\tau))d\tau\Big]dt\right].$$

Now using the definition of generalized convolution definition

$$\Rightarrow F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = p(s)\left[p(s)\int_0^\infty \epsilon'(t)(f*g)(\epsilon'(t))e^{-q(s)\epsilon(t)}dt\right]$$

$$\Rightarrow \frac{1}{p(s)}F_{\epsilon(t)}(s)G_{\epsilon(t)}(s) = \Im_{\epsilon(t)}\{[f*g](\epsilon(t));s\}.$$

Hence the proof. □

Theorem 4 (**Second Derivative Test**). *Let $f(t)$ satisfies all the conditions given in Manjarekar (2017) with $f(t)$ is differentiable function and $p(s), q(s)$ are positive real valued functions then;*

$$\Im_{\epsilon(t)}\{f''(t);s\} = q(s)\Bigg[q(s)\Im_{\epsilon(t)}\{(\epsilon'(t))^2 f(t);s\} - p(s)\int_0^\infty \epsilon''(t)f'(t)e^{-q(s)\epsilon(t)}dt$$
$$- p(s)f'(0)\epsilon'(0^+)e^{-q(s)\epsilon(0^+)}\Bigg] - p(s)\int_0^\infty \epsilon(t)f(t)e^{-q(s)\epsilon(t)}dt - p(s)f'(0)e^{-q(s)\epsilon(0^+)}.$$

Proof. By the definition of modified general integral transform (4), put $h(t) = f'(t) \Rightarrow h'(t) = f''(t)$ and apply (Theorem 2)

$$\Im_{\epsilon(t)}\{h'(t);s\} = q(s)[\Im_{\epsilon(t)}\{\epsilon'(t)h(t);s\}] - p(s)\int_0^\infty \epsilon''(t)h(t)e^{-q(s)\epsilon(t)}dt - p(s)h(0)\epsilon'(0^+)e^{-q(s)\epsilon(0^+)},$$

which finally leads to our result.

$$\Im_{\epsilon(t)}\{f''(t);s\} = q(s)\Bigg[q(s)\Im_{\epsilon(t)}\{(\epsilon'(t))^2 f(t);s\} - p(s)\int_0^\infty \epsilon''(t)f'(t)e^{-q(s)\epsilon(t)}dt$$
$$- p(s)f'(0)\epsilon'(0^+)e^{-q(s)\epsilon(0^+)}\Bigg] - p(s)\int_0^\infty \epsilon(t)f(t)e^{-q(s)\epsilon(t)}dt - p(s)f'(0)e^{-q(s)\epsilon(0^+)}.$$

The above property coincides with Laplace Transform property by taking $p(s) = 1$ and $q(s) = s$ along with $\epsilon(t) = t$. In the similar manner using induction one can find integral transform of $f^{(n)}(t)$ at a point 's' satisfying the required conditions. □

Theorem 5 (**Multiplication by $\epsilon(t)$**). *Let $f(t)$ satisfies all the conditions given in Manjarekar (2017) with $f(t)$ is differentiable function while $p(s), q(s)$ are positive real valued functions then;*

$$\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\} = -\frac{p(s)}{q'(s)}\left(\frac{\Im_{\epsilon(t)}\{f(t);s\}}{p(s)}\right)'(s).$$

Proof. By definition of modified general integral transform (4) we have

$$\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\} = p(s)\int_0^\infty \epsilon'(t)f(t)\epsilon(t)e^{-q(s)\epsilon(t)}dt$$

$$\frac{\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\}}{p(s)} = \int_0^\infty \epsilon'(t)f(t)\epsilon(t)e^{-q(s)\epsilon(t)}dt.$$

Differentiating both sides of the above equation w.r.t. s, it yields into

$$\frac{d}{ds}\left(\frac{\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\}}{p(s)}\right) = -q'(s)\int_0^\infty \epsilon'(t)f(t)\epsilon(t)e^{-q(s)\epsilon(t)}dt.$$

Multiplying by $p(s)$ on both sides of the equation above

$$p(s)\frac{d}{ds}\left(\frac{\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\}}{p(s)}\right) = -q'(s)p(s)\int_0^\infty \epsilon'(t)f(t)\epsilon(t)e^{-q(s)\epsilon(t)}dt.$$

Rearranging the terms ives the final result as

$$\Im_{\epsilon(t)}\{\epsilon(t)f(t);s\} = -\frac{p(s)}{q'(s)}\left(\frac{\Im_{\epsilon(t)}\{f(t);s\}}{p(s)}\right)'(s).$$

The above property overlaps with Laplace transform taking $p(s) = 1$ and $q(s) = s$ with $\epsilon(t) = t$ which gives us $\Im_{\epsilon(t)}\{tf(t);s\} = -F'(s)$
By using induction, we get;

$$\Im_{\epsilon(t)}\{[\epsilon(t)]^n f(t);s\} = (-1)^n \frac{p(s)}{q'(s)}\left(\frac{1}{[q'(s)]^n}\frac{d^{(n)}}{ds^{(n)}}\frac{\Im_{\epsilon(t)}\{f(t);s\}}{p(s)}\right)(s).$$

□

Theorem 6. *Let $f(t)$ satisfies all the conditions given in Manjarekar (2017) with $f(t)$ is differentiable n – times and $p(s), q(s)$ are positive real valued functions then;*

$$\Im_{\epsilon(t)}\{\epsilon(t)f^{(n)}(t);s\} = -p(s)q'(s)\frac{d}{ds}\left[\frac{1}{p(s)}\Im_{\epsilon(t)}\{f^{(n)}(t);s\}\right]$$

Proof. By definition of modified general integral transform (4) we have

$$\Im_{\epsilon(t)}\{\epsilon(t)f^{(n)}(t);s\} = p(s)\int_0^\infty \epsilon'(t)\epsilon(t)f^{(n)}(t)e^{-q(s)\epsilon(t)}dt$$

rearranging the terms

$$\Rightarrow \Im_{\epsilon(t)}\{\epsilon(t)f^{(n)}(t);s\} = -p(s)q'(s)\int_0^\infty \frac{d}{ds}\left[\epsilon'(t)f^{(n)}(t)e^{-q(s)\epsilon(t)}\right]dt.$$

Which finally gives

$$\Im_{\epsilon(t)}\{\epsilon(t)f^{(n)}(t);s\} = -p(s)q'(s)\frac{d}{ds}\left[\Im_{\epsilon(t)}\{f^{(n)}(t);s\}\right].$$

Hence the proof □

4 Relationship with other Integral Transforms

In this section, we compare the with other integral transform. In view of (2) and (4), it is clear that when $\epsilon(t) = t$ then the modified general integral transform is equivalent general integral transform, $(\Im_{\epsilon(t)} f(t); s\} = T\{f(t); s\} = F(s))$. As it is shown in the Section 3 of Jafari (2021), the Laplace, α-Laplace, Sawi, Elzaki, Sumudu, Natural, Aboodh, Pourreza, Mohand, G_transform and Kamal transforms are special case of general integral transform, it is easy to show under same assumption, we have same result. Therefor we just compare here this transform with Laplace - Carson, L_2, Mellin and Abel transforms.

- **Relationship with Laplace - Carson Transform**
  The Laplace - Carson Transformation Jafari (2021) of $f(t)$ were obtained by substitution of $p(s) = s, q(s) = s$ with $\epsilon(t) = t$ in (4) with $Re(s) > 0$.

- **Relationship with L_2 Transform**
  The L_2 Transform Yurekli (1999) of $f(t)$ were obtained by substitution of $p(s) = \frac{1}{2}, q(s) = s^2$ with $\epsilon(t) = t^2$ in (4) with $Re(s^2) > 0$.

$$\Im_{\epsilon(t)} f(t); s\} = p(s) \int_0^\infty \epsilon'(t) f(t) e^{-q(s)\epsilon(t)} dt$$

$$\Rightarrow \Im_{t^2} f(t); s\} = \frac{1}{2} \int_0^\infty 2t f(t) e^{-ts^2} dt$$

$$\Rightarrow \Im_{t^2} f(t); s\} = \int_0^\infty t f(t) e^{-ts^2} dt.$$

  Using the relationship of Laplace - Carson and Laplace - Stieltjes Transform and the definition of new modified general integral transformation we have; $\Im_t f(t); s\} = \Im_t f'(t); s\}$

- **Relationship with Mellin Transform**
  The Mellin Transform Zhao (2015) of $f(t)$ were obtained by substitution of $p(s) = 1, q(s) = -s$ with $\epsilon(t) = \ln(t)$ in (4) with $Re(s) > 0$.

$$\Im_{\epsilon(t)} f(t); s\} = p(s) \int_0^\infty \epsilon'(t) f(t) e^{-q(s)\epsilon(t)} dt$$

$$\Rightarrow \Im_{\ln(t)} f(t); s\} = \int_0^\infty \frac{1}{t} f(t) e^{s\ln(t)} dt$$

$$\Rightarrow \Im_{\ln(t)} f(t); s\} = \int_0^\infty \frac{1}{t} f(t) t^s dt$$

$$\Rightarrow \Im_{\ln(t)} f(t); s\} = \int_0^\infty t^{s-1} f(t) dt.$$

- **Relationship with Abel's Transform**

  The Abel's Transform Rudolf (2006) of $f(t)$ were obtained by substitution of $p(s) = 2, q(s) = -1$ with $\epsilon(t) = \ln(\sqrt{(t^2 - y^2)})$ for $t > y$ and 0 otherwise in (4) .

$$\Im_{\epsilon(t)} f(t); s\} = 2 \int_y^\infty \frac{1}{(\sqrt{(t^2 - y^2)}} 2t f(t) e^{\ln(\sqrt{(t^2-y^2)})} dt$$

$$\Im_{\epsilon(t)} f(t); s\} = 2 \int_y^\infty \frac{1}{\sqrt{(t^2 - y^2)}} \frac{1}{2\sqrt{(t^2 - y^2)}} 2t f(t) (\sqrt{(t^2 - y^2)} dt$$

$$\Im_{\epsilon(t)} f(t); s\} = \int_y^\infty \frac{2t}{\sqrt{(t^2 - y^2)}} f(t) dt.$$

  Which Abel's Transform under the given conditions.

5 Application towards solution of ODE and Integral Equations

In this section we use modified general integral transform to solve initial value problems with variable coefficients and constant coefficients also Volterra integral equation and fractional differential equations.It is easy to show that for $\epsilon(t) = t$ the equation (4) becomes the new general integral transform (Jafari, 2021) using which one can find the solutions towards the IVP, ordinary differential and Volterra integral equations.

1. Consider the second order ordinary differential equation

$$y'' - 4y = 0,$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$.

By applying modified general integral transformation on both sides of the above equation with $\epsilon(t) = t$ and using the property for modified general integral transformation of First order derivative gives;

$$\Im_{\epsilon(t)} y''(s); s\} - 4\Im_{\epsilon(t)} y(t); s\} = 0$$

$$\Rightarrow q^{(2)}(s)\Im_{\epsilon(t)} y(s); s\} - p(s)\Big[q(s)y(0) + y'(0)\Big] - 4\Im_{\epsilon(t)} y(s); s\} = 0.$$

Using initial conditions, it gives us,

$$\Rightarrow q^{(2)}(s)\Im_{\epsilon(t)} y(s); s\} - p(s)q(s) - 4\Im_{\epsilon(t)} y(s); s\} = 0.$$

Rearranging the terms gives

$$\Im_{\epsilon(t)} y(s); s\} = \frac{p(s)q(s)}{q^{(2)}(s) - 4} = \frac{p(s)q(s)}{(q(s) - 2)(q(s) + 2}.$$

By the method of partial fraction it resulted into

$$\Im_{\epsilon(t)} y(s); s\} = \frac{1}{2}\frac{p(s)q(s)}{(q(s) - 2)} + \frac{1}{2}\frac{p(s)q(s)}{(q(s) + 2)}.$$

Taking inverse new general integral transform (Jafari, 2021). We get the solution as; $y(t) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$, which is desired solution.

Now, moving towards the solution of Volterra integral equation one has to use convolution Theorem proved above;

2. Consider the Volterra integral equation given as follows:

$$y(t) = 1 + \int_0^t (t - \tau)y(\tau)d\tau.$$

By applying modified general integral transform on both sides of the above equation with $\epsilon(t) = t$ and using the property for modified general integral transform of convolution gives;

$$
\begin{aligned}
\Im_t[y(t);s] &= \Im_t\{1;s\} + \Im_t\left[\int_0^t (t-\tau)y(\tau)d\tau;s\}\right] \\
\Rightarrow \Im_t[y(t);s] &= \frac{p(s)}{q(s)} + \frac{1}{p(s)}\frac{p(s)}{[q(s)]^2}\Im_t y(t);s\} \\
\Rightarrow & \quad \left[1-\frac{1}{p(s)}\frac{p(s)}{[q(s)]^2}\right]\Im_t[y(t);s] = \frac{p(s)}{q(s)} \\
\Rightarrow & \quad \Im_t[y(t);s] = \frac{1}{2}\left[\frac{p(s)q(s)}{q(s)-1}\right] + \frac{1}{2}\left[\frac{p(s)q(s)}{q(s)+1}\right].
\end{aligned}
$$

Taking the inverse general integral transform on both side; we get solution as (Jafari, 2021)

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t},$$

which is required solution of the given integral equations.

3. We consider the Linear partial fractional differential equation with constant coefficients (Ansari, 2012) given as follows:

$$^{RL}D^{\alpha}_{0+}y(x,t) = -y_x(x,t),$$

where $0 < \alpha \leq 1$ and $D^{\alpha-1}_{0+} = f(x)$ along with $y(0,t) = 0$ under the condition that $x, t > 0$

To find the solution of this equation having Cauchy type initial and boundary conditions; we define $\epsilon(x)$ as follows:

$$\epsilon(x) = \int e^{-\frac{1}{2}\ln(1+x)} = 2\sqrt{1+x}.$$

Apply this $\Im_{2\sqrt{1+x}}$ - transformation in space domain and the Laplace transform in time domain as follows:

$$\Im_x[y(x,t);s] = \tilde{y}(x,s) = \textstyle\int_0^\infty y(x,t)e^{-st}dt.$$

$$\Im_{2\sqrt{1+x}}[y(x,t);s] = \textstyle\int_0^\infty \frac{1}{\sqrt{1+x}}y(x,t)e^{-2s\sqrt{1+x}}dx = \hat{y}(s,t).$$

Now by using the Laplace transform of the R - L fractional derivative [Ansari (2012)] and $\Im_{2\sqrt{1+x}}[y(x,t);s]$ - transform of the given equation

$$\Im_x[^{RL}D^{\alpha}_{0+}[y(x,t);s]] = s^{\alpha}\tilde{y}(x,s) -^{RL} D^{\alpha-1}_{0+}[y(x,0^+)]$$

$$\Im_{2\sqrt{1+x}}[y_x(x,t);p] = p\hat{y}(p,t) - y(0,t).$$

After rearranging the terms and using the Cauchy type initial conditions; it gives us

$$y(\hat{\hat{p}},s) = \tfrac{1}{S^{\alpha}+p}F(p),$$

where $F(p)$ is the $\Im_{2\sqrt{1+x}}$ - transformation of the initial condition $f(x)$ and applying (1) and (3) i.e. inversion and convolution theorem gives;

$$\begin{gathered}\tilde{y}(x,s) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y(\hat{\hat{p}},s)e^{-2p\sqrt{1+x}}dp \\ \Rightarrow \tilde{y}(x,s) = \Im^{-1}_{\sqrt{1+x}}[(\frac{1}{s^{\alpha}+p}) * f(x)] \\ \Rightarrow \tilde{y}(x,s) = e^{-2\sqrt{1+x}s^{\alpha} * f(x)},\end{gathered}$$

where the convolution of the above two functions for $\Im_{2\sqrt{1+x}}$ - transformation can be expressed as [Ansari (2012)];

$$[f * g] = \int_0^x g(t)f\left(\frac{2\sqrt{1+x}-2\sqrt{1+t}}{\sqrt{1-(2\sqrt{1+x}-2\sqrt{1+t})^2}}\right)dt.$$

Now in regards to the inverse Laplace transform of the function $e^{-2\sqrt{1+x}s^{\alpha}}$ for $\alpha = 0.5$ as

$$\Im_x^{-1} e^{-2\sqrt{1+x}s^{0.5}} = \frac{\sqrt{1+x}e^{\frac{(x+1)}{t}}}{\sqrt{\pi}t^{\frac{3}{2}}},$$

provided $\sqrt{1+x} > 0$ So that the solution of this equation is

$$y(x,t) = \int_0^x G^{0.5}(2\sqrt{1+x} - 2\sqrt{1+\tau})f(\tau)d\tau.$$

In terms of Green's function, provided the integral on r.h.s is convergent.

4. Consider another Linear partial fractional differential equation with constant coefficients (Ansari, 2012) given as follows:

$$^{RL}D^{\alpha}_{0^+}y(x,t) = -\frac{1}{a}y_x(x,t),$$

where, $a > e$,$0 < \alpha \leq 1$ and $D^{\alpha-1}_{0^+} = f(x)$ along with $y(0,t) = 0$ under the condition that $x, t > 0$

To find the solution of this equation having Cauchy type initial and boundary conditions; we define $\epsilon(x)$ as follows:

$$\epsilon(x) = \int e^{-\ln(\frac{1}{a})} = ax + b$$

where, $a > e$ along with $p(s) = \frac{1}{a}$ and $q(s) = \frac{s}{a}$ Apply this $\Im_{ax+b}$ - transformation in space domain and the Laplace transform in time domain as follows:

$$\Im_x[y(x,t);s] = \tilde{y}(x,s) = \int_0^{\infty} y(x,t)e^{-st}dt$$

$$\Im_{ax+b}[y(x,t);s] = \int_0^{\infty} p(s)\epsilon'(x)y(x,t)e^{-q(s)\epsilon(x)}dx = \hat{y}(s,t)$$

$$\Im_{ax+b}[y(x,t);s] = \int_0^{\infty} \frac{1}{a}ay(x,t)e^{-[\frac{s}{a}(ax+b)]}dx.$$

This can be rewritten in terms of Laplace transformation as;

$$\Im_{ax+b}[y(x,t);s] = e^{\frac{-bs}{a}}L[y(x,t);s],$$

which gives us;

$$L[y(x,t);s] = e^{\frac{bs}{a}} \Im_{ax+b}[(ax+b)y(x,t);s].$$

Now by using the Laplace transform of the R-L fractional derivative (Ansari, 2012) and $\Im_{ax+b}[y(x,t);s]$ - transform of the given equation

$$\Im_x[{}^{RL}D^{\alpha}_{0^+}[y(x,t);s]] = s^{\alpha}\tilde{y}(x,s) -{}^{RL} D^{\alpha-1}_{0^+}[y(x,0^+)]$$

$$\Im_{ax+b}[y_x(x,t);s] = e^{\frac{-bs}{a}} L[y_x(x,t);s] = e^{\frac{-bs}{a}}[s\hat{y}(s,t) - y(0,t)]$$

which turns out to be

$$s^{\alpha}\hat{y}(s,t) - F(s) = -se^{\frac{-bs}{a}}\hat{y}(s,t) \quad a > e$$

After re - arranging the terms it gives us;

$$\hat{y}(s,t) = \frac{F(s)}{se^{-\frac{bs}{a}} + s^{\alpha}}.$$

By taking the inverse Laplace transform which is equivalent to take $\Im^{-1}_{ax+b}$ under the condition that $a > e$ alongwith $p(s) = \frac{1}{a}$ and $q(s) = \frac{s}{a}$ it gives the solution in terms of convolution as;

$$y(x,t) = f(x) * L^{-1}\left(\frac{1}{se^{-\frac{bs}{a}} + s^{\alpha}}\right),$$

under the condition $a > e$.

6 Conclusion

The paper gives some new ideas in the field of generalized integral transformations as well as it might extend to an application in the field of fractional calculus. Also, the newly defined modified new general integral transform gives relationship with other integral transformations. We can accomplish that by selection of proper choice of $p(s), q(s)$ and $\epsilon(t)$ the ordinary differential equation can be solved using modified generalized integral transformation. Thus from modified generalized transformation various Transforms can be obtained by imposing different condition on it which helps to find out the solution of ordinary differential equation and Volterra integral equations which might be extend to find out the solution of fractional differential equations.

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